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Anti-de Sitter space and the center of the gauge group

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Abstract: Upon compactification on a circle, $SU(N)$ gauge theory with all fields in the adjoint representation acquires a $\mathbf{Z}_N$ global symmetry because the center of the gauge group is $\mathbf{Z}_N$. For $\mathcal{N}=4$ super Yang-Mills theory, we show how this $\mathbf{Z}_N$ *topological symmetry* arises in the context of the AdS/CFT correspondence, and why the symmetry group is $\mathbf{Z}_N$ rather than $U(1)$. This provides a test of the AdS/CFT correspondence for finite N. If the theory is formulated on $\mathbf{R}^3 \times \mathbf{S}^1$ with anti-periodic boundary conditions for fermions around the $\mathbf{S}^1$, the topological symmetry is spontaneously broken; we show that the domain walls are D-strings, and hence that flux tubes associated with magnetic confinement can end on the domain walls associated with the topological symmetry. For the $(0,2)$ A_{N-1} superconformal field theory in six dimensions, we demonstrate an analogous phenomenon: a $\mathbf{Z}_N$ global symmetry group arises if this theory is compactified on a Riemann surface. In this case, the domain walls are M-theory membranes.

Keywords: Field Theories in Higher Dimensions, Gauge Symmetry, Discrete and Finite Symmetries, Global Symmetries.

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1. Introduction

According to a by now celebrated conjecture [1], $\mathcal{N}=4$ super Yang-Mills theory with $SU(N)$ gauge group on $\mathbf{S}^4$ is equivalent to Type IIB superstring theory on $AdS_5\times\mathbf{S}^5$, with N units of five-form flux on $\mathbf{S}^5$, and with the complex coupling constant τ_{IIB} of Type IIB string theory identified with the Yang-Mills coupling constant $\tau_{YM}=\frac{\theta}{2\pi}+\frac{4\pi i}{g_{YM}^2}$. Correlation functions of the gauge theory are expressed [2, 3] in terms of the dependence of the Type IIB string theory on boundary conditions.

The manifold $\mathbf{S}^4$ (which can be viewed as the conformal compactification of $\mathbf{R}^4$) enters because it is the boundary, in a conformal sense, of AdS_5; to study the $\mathcal{N}=4$ theory on another four-manifold M, one would replace AdS_5 by a negatively curved Einstein manifold of boundary M. An interesting example is to take $M_4=M_3\times\mathbf{S}^1$, with M_3 a three-manifold. If one uses antiperiodic boundary conditions for fermions in the $\mathbf{S}^1$ direction, then quantum field theory on M_4 is related to a thermal ensemble: it computes $\mathrm{Tr}\,e^{-\beta H}$ with H the Hamiltonian in quantization on M_3. In the present paper, we concentrate on the cases that M_3 is $\mathbf{S}^3$ or $\mathbf{R}^3$.

The $\mathcal{N}=4$ super Yang-Mills theory has all fields in the adjoint representation of $SU(N)$. So, if this theory is formulated on $M_3\times\mathbf{S}^1$, it develops a $\mathbf{Z}_N$ global symmetry that has the following origin. Let $\widetilde{G}_0$ be the group of $SU(N)$ gauge transformations on $M_3\times\mathbf{S}^1$, and let $\widetilde{G}$ be the group of $SU(N)/\mathbf{Z}_N$ gauge transformations on the same manifold. $\widetilde{G}$ includes gauge transformations by gauge functions $g: M_3\times\mathbf{S}^1\to SU(N)/\mathbf{Z}_N$ which, if lifted to $SU(N)$, would be multiplied in going around the $\mathbf{S}^1$ by an element of the center of $SU(N)$. In quantizing $SU(N)$ gauge theory on $M_3\times\mathbf{S}^1$, one divides by $\widetilde{G}_0$, and the quotient

$$\Gamma=\widetilde{G}/\widetilde{G}_0 \tag{1.1}$$

is observed as a global symmetry. Γ is isomorphic to the center of $SU(N)$, which in turn is naturally identified with the group of N^{th} roots of unity. We will call any symmetry such as Γ which depends on the topology of spacetime a topological symmetry.[1] An order parameter for the $\mathbf{Z}_N$ global symmetry is a *temporal Wilson line.* Let P be a point in M_3, let $C_P = P \times \mathbf{S}^1$, and let

$$W(P) = \mathrm{Tr}\, P \exp \oint_{C_P} A\,, \tag{1.2}$$

with the trace taken in the fundamental representation of $SU(N)$. A gauge transformation which in wrapping around the circle is multiplied by an N^{th} root of unity ζ multiplies $W(P)$ by ζ. Therefore, the expectation value of $W(P)$ can serve as an order parameter for the $\mathbf{Z}_N$ symmetry; if Γ is unbroken, expectation values

$$\langle W(P_1)W(P_2)\dots W(P_k)\rangle \tag{1.3}$$

vanish unless k is divisible by N.

We have so far assumed that the gauge group is $SU(N)$ rather than $U(N)$; for $U(N)$ gauge theory, the symmetry group is $U(1)$ rather than $\mathbf{Z}_N$. There are indeed strong arguments which show that Type IIB on $AdS_5 \times \mathbf{S}^5$ is related to $SU(N)$ rather than $U(N)$ gauge theory [3]. But there is also a puzzle [4], since a study of a high temperature phase on $\mathbf{S}^3 \times \mathbf{S}^1$ appeared to show a $U(1)$ topological symmetry group. We resolve this question in section 2. We show that on $\mathbf{S}^3$ (and similarly on any three-manifold M_3), the topological symmetry group at either high temperatures or low temperatures is $\mathbf{Z}_N$, not $U(1)$. However, to show this at high temperatures requires a novelty involving a certain spacetime θ angle.

Since our question probes the fact that the center of the gauge group is $\mathbf{Z}_N$ rather than $U(1)$, it is naturally closely related to the existence of a baryon vertex (the possibility of forming a gauge-invariant combination of Nexternal quarks), which probes the same issue. There have been two related but somewhat different proposals concerning the baryon vertex: it has been constructed from a wrapped fivebrane [5], or connected with the existence of a certain low energy coupling $B_{NS} \wedge H_{RR} \wedge G_5$ [6]. As we will see, each of these facets of the baryon vertex is helpful in understanding the topological symmetry; the first is more directly relevant at low temperatures, and the second at high temperatures.

The identification of the topological symmetry group is one of a few direct tests of the conjecture of [1] for finite values of N. Many of the direct tests of the conjecture up to now test the conjecture only in the limit $N \to \infty$.

We also consider the case that M_3 is decompactified to $\mathbf{R}^3$. Thus, in this case we are studying the $\mathcal{N} = 4$ theory on $\mathbf{R}^3 \times \mathbf{S}^1$. In this infinite volume situation, the topological symmetry group is spontaneously broken, as already shown from the Anti-de Sitter point of view in [4]. Since the topological symmetry is a discrete symmetry,

[1] The full topological symmetry group of $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory compactified on a circle is actually $\mathbf{Z}_N \times \mathbf{Z}_N$, as discussed at the end of section 2.

its breaking results in the existence of domain walls. One might think that the domain walls would be solitonic objects in the large N limit, with tensions of order N^2 (the large N effective theory of glueballs has an effective action proportional to N^2, and all solitonic objects would have energy or energy density of order N^2). However, it turns out that the domain walls are in fact D-strings, with tensions of order N. Such behavior is by now familiar from various examples, some of them treated in [5], where it has turned out that various particles, strings, or domain walls of large N gauge theory, which one might think would be solitonic objects of the large N effective theory, are better understood as D-branes.

In section 3, we move on to discuss the $(0,2)$ theory in six dimensions. The A_{N-1} version of this theory is constructed, as in [1], via its conjectured dual which is the compactification of M-theory on $AdS_7 \times \mathbf{S}^4$, with N units of four-form flux on $\mathbf{S}^4$. If this theory is compactified on a circle, it looks at low energies like $SU(N)$ Yang-Mills theory, and hence may be expected, if further compactified on a second circle, to exhibit a $\mathbf{Z}_N$ topological symmetry. Thus, if the $(0,2)$ theory is compactified on $\mathbf{S}^1 \times \mathbf{S}^1 = \mathbf{T}^2$, one should expect a $\mathbf{Z}_N$ global symmetry. In section 3, we establish a stronger result: we show that if the $(0,2)$ theory is compactified on a Riemann surface F of any genus, not necessarily $\mathbf{T}^2$, it develops a $\mathbf{Z}_N$ symmetry. This assertion cannot at present be tested independently in any obvious way. It seems like an interesting probe of the inner nature of the still rather mysterious $(0,2)$ theory.

2. AdS space and the topological symmetry of YM theory

2.1. Behavior on compact spacetime

We will mainly consider $\mathcal{N}=4$ super Yang-Mills on $\mathbf{S}^3 \times \mathbf{S}^1$. This manifold is [7] the boundary of two different negatively curved Einstein manifolds X_1 and X_2:

(1) X_1 is a quotient of AdS_5 and has topology $D_4 \times \mathbf{S}^1$, where D_4 is a four-ball of boundary $\mathbf{S}^3$.

(2) X_2 is the Euclidean Schwarzschild black hole solution of Anti-de Sitter space; its topology is $\mathbf{S}^3 \times D_2$, where D_2 is a disc or two-ball of boundary $\mathbf{S}^1$.

$\mathcal{N}=4$ super Yang-Mills on $\mathbf{S}^3$ at low or high temperatures, respectively, is described (for large N and $g_{YM}^2 N$) by Type IIB superstrings on $X_1 \times \mathbf{S}^5$ or $X_2 \times \mathbf{S}^5$ [4].

Now we consider, as explained in the introduction, the problem of computing an expectation value of a product of k temporal Wilson lines:

$$\langle W(P_1) \dots W(P_k) \rangle \,. \tag{2.1}$$

The P_i are points in $\mathbf{S}^3$, and the i^{th} Wilson line is wrapped on $C_i = P_i \times \mathbf{S}^1$. According to [8, 9], this expectation value equals a string theory partition function in which one sums over configurations in which the C_i are boundaries of elementary string worldsheets Σ_i.[2]

Let us first understand what happens in the low temperature phase. Here, we seem to have a problem: the C_i are not boundaries of any closed manifold in $X_1 \times \mathbf{S}^5$, so naively all the expectation values (2.1) vanish. Luckily the baryon vertex, as described in [5], comes to the rescue. We wrap a D5-brane on $V = Q \times \mathbf{S}^1 \times \mathbf{S}^5$, where Q is any point in D_4. We now meet a phenomenon explained in [5]: because of the fiveform flux on $\mathbf{S}^5$, a fivebrane can be wrapped on V only if elementary string worldsheets whose boundaries have total winding number N around the $\mathbf{S}^1$ end on V. We take these boundaries to consist of N circles $D_i \subset V$ which each wrap around $\mathbf{S}^1$ once. If $k = N$, we connect each C_i to D_i by an elementary string worldsheet Σ_i; this is possible because C_i and D_i have the same winding number around the circle. This configuration with a fivebrane connected to the circles C_i by elementary string worldsheets gives a nonvanishing contribution to (2.1) for $k = N$. For k any multiple of N, one makes a similar configuration with several wrapped fivebranes; for k not a multiple of N, there are no such contributions. This shows, from the point of view of the low temperature phase, that the topological symmetry is $\mathbf{Z}_N$.

Now we consider the high temperature phase, which is described by Type IIB on $X_2 \times \mathbf{S}^5$. Here we seem to have an opposite puzzle, which is that the circles C_i are all boundaries in X_2; in fact, since D_2 has boundary $\mathbf{S}^1$, C_i is the boundary of $P_i \times D_2$. Now the expectation value $\langle W(P)\rangle$ of a single temporal loop on $P \times \mathbf{S}^1$ seems to receive a contribution from a path integral with an elementary string worldsheet of the form $\Sigma = P \times D_2 \times R \subset X_2 \times \mathbf{S}^5$ (R is a point in $\mathbf{S}^5$ which actually enters, as mentioned in the footnote 2, in the detailed definition of $W(P)$). A nonzero expectation value

$$\langle W(P)\rangle \tag{2.2}$$

would contradict the existence of the topological symmetry.

This problem was partly resolved in [4] by noticing that, with no cost in energy, we can turn on an NS two-form field B_{NS} with $dB_{NS} = 0$ and an arbitrary value of

$$\alpha = \int_{D_2} B_{NS}\,. \tag{2.3}$$

Because of the coupling of B_{NS} to the elementary string worldsheet, the contribution described in the last paragraph to the expectation value (2.2) is proportional to $e^{i\alpha}$.

If there is a symmetry under

$$\alpha \to \alpha + \text{constant}\,, \tag{2.4}$$

[2] Actually, to be more exact, the operators whose expectation values one computes this way are not conventional Wilson lines, but receive also contributions from the scalar fields. The contribution of the scalars is determined, for each $W(P_i)$, by the choice of a point $R_i \in \mathbf{S}^5$; the boundary of Σ_i is required to be $C_i \times R_i$. For details, see [8, 9]. As much as possible, we will suppress the R_i in the discussion.

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then the expectation value in question will vanish upon averaging over α (recall that we are discussing the theory at finite volume, so we should sum over all possible values of α). To get the desired result that (2.2) vanishes, but (2.1) is nonzero if k is a multiple of N, we want to have a symmetry not under arbitrary shifts of α, but only under

$$\alpha \to \alpha + \frac{2\pi s}{N}\,, \qquad s = 0, \ldots, N-1\,. \tag{2.5}$$

(To be more precise, (2.5) will turn out to be a symmetry for any integer s, but if s is divisible by N, it is equivalent to a gauge transformation that is trivial at spatial infinity and so acts as the identity on all observables.) Invariance under arbitrary translations of α would correspond to an unwanted $U(1)$ topological symmetry group.

What effects in the theory do not have the invariance (2.4)? Any term in a low energy effective Lagrangian which is the integral of a gauge-invariant local density will involve the B_{NS} field only via $H_{NS} = dB_{NS}$. So such terms are independent of α.

A term in the Type IIB low energy effective action which cannot be written as the integral of a gauge-invariant local density is

$$\Delta L = -\int B_{NS} \wedge \frac{H_{RR}}{2\pi} \wedge \frac{G_5}{2\pi}\,. \tag{2.6}$$

Here $H_{RR} = dB_{RR}$ is the field strength of the Ramond-Ramond two-form field B_{RR}; G_5 is the fiveform field strength. (We have written the interaction for a spacetime of Lorentz signature; in Euclidean signature, the minus sign is replaced by a factor of i.) This interaction is a likely candidate for solving our problem, since [6] it is intimately related to the fivebrane wrapping mode which solves the problem in the low temperature regime.[3]

Now, in working on the spacetime $X_2 \times \mathbf{S}^5$ which is topologically $\mathbf{S}^3 \times D_2 \times \mathbf{S}^5$, we can consider an *instanton* in which H_{RR} is pulled back from the $\mathbf{S}^3$ factor and

$$\int_{\mathbf{S}^3} \frac{H_{RR}}{2\pi} = s\,, \tag{2.7}$$

with any desired integer s. (An important detail is that the behavior of the metric of X_2 near infinity is such that such a field has finite action despite the noncompactness of D_2.) We also have

$$\int_{\mathbf{S}^5} \frac{G_5}{2\pi} = N\,. \tag{2.8}$$

We assume that B_{NS} is pulled back from D_2 and obeys (2.3). Then, we can evaluate

$$\Delta L = -Ns\alpha\,. \tag{2.9}$$

[3] Actually, because the self-dual fiveform field is non-Lagrangian, there is a subtlety in the claim that the Type IIB theory has the interaction ΔL. One precise and true statement is that the equations of motion for B_{NS} and B_{RR} contain the terms one would expect by varying ΔL. For our purposes, one can think of G_5 as a fixed background field, and then the fields B_{NS}, B_{RR} can be described by an effective Lagrangian that contains the term ΔL. Alternatively, we can write down an effective 5-dimensional Lagrangian for the Type IIB theory compactified on $\mathbf{S}^5$, and it will contain the terms which arise from integrating (2.6) on $\mathbf{S}^5$.

The factor $\exp(i\Delta L)$ in the path integral thus equals $\exp(-iNs\alpha)$. The $s = 1$ contribution, for example, is invariant precisely under the desired topological symmetry group (2.5).

Now, in evaluating (2.1), we get a factor of $e^{i\alpha}$ from each string world-sheet bounded by one of the C_i; this makes a factor of $\exp(ik\alpha)$ in all. To get a nonzero result after including the instanton factor arising from s instantons and integrating over α, we need $\exp(ik\alpha - iNs\alpha) = 1$, or $k = Ns$. Thus, a nonzero contribution can arise, for some s, if and only if k is a multiple of N. This shows that the topological symmetry is $\mathbf{Z}_N$ also in the high temperature phase.

2.2. Infinite volume limit

Now we would like to take the infinite volume limit, in the three-dimensional sense, and replace $\mathbf{S}^3$ by $\mathbf{R}^3$. In the infinite volume limit the finite temperature theory is always in the high temperature phase, so we consider Type IIB superstring theory on $Z = \mathbf{R}^3 \times D_2 \times \mathbf{S}^5$.

In this limit, one cannot naturally compute using instantons of the H_{RR} field, because there are no such localized instantons; a field with a non-zero value of $\int_{\mathbf{R}^3} H_{RR}$ will tend to spread out so as to minimize the action. Nevertheless, since fields with a nonzero integral of H_{RR} gave the essential contribution in finite volume, it is fairly clear that treating such fields quantum mechanically must be the key also in infinite volume.

The appropriate ideas can be found by a fairly straightforward adaptation of the analysis of $U(1)$ gauge theory in two dimensions [10]. We write simply B and H for the components of B_{RR} and H_{RR} that are pulled back from the $\mathbf{R}^3$ factor in Z. At $\alpha = 0$, the H field has simply a quadratic Lagrangian of the form

$$L_H = \frac{1}{12e^2}\int_{\mathbf{R}^3} d^3x \; H_{IJK}H^{IJK} \tag{2.10}$$

plus higher order terms (more derivatives or higher powers of H) that will be inessential for our purposes. Here $I, J, K = 1, 2, 3$, and e is a constant, independent of N, that is obtained by evaluating the ten-dimensional action for the field H that is pulled back from $\mathbf{R}^3$. (Note that $1/e^2$ is finite, despite the noncompactness of $D_2 \times \mathbf{S}^5$.)

To quantize the theory, we split the coordinates as a time coordinate 0 and space coordinates $I = 1, 2$. We work in a gauge in which the only nonzero component of B is B_{12}. A quantum state must be a gauge-invariant function of B (invariant, that is, under $B \to B + d\Lambda$, where $d\Lambda$ is a closed two-form with periods that are integer multiples of 2π). A basis of states with this property is furnished by

$$\Psi_r = \exp\left(ir\int_{\mathbf{R}^2} B\right), \tag{2.11}$$

with $r \in \mathbf{Z}$. The physical interpretation of Ψ_r for $r \neq 0$ is that it describes a state with H_{012} non-zero. To show this, one notes that if $\Pi(x) = -i\delta/\delta B(x)$ is the canonical momentum to B, then from the explicit form of Ψ_r we have

$$\Pi(x)\Psi_r = r\Psi_r \tag{2.12}$$

for all x. On the other hand, since $\Pi(x) = \delta L_H/\delta\partial_0 B(x)$, we get $\Pi = H_{012}/e^2$. So the state Ψ_r is characterized by $H_{012} = e^2 r$. Note that this constant value of H_{012} is completely invariant under the connected component of the three-dimensional Poincaré group, though it is odd under parity. Since the Hamiltonian density is

$$\mathcal{H} = \frac{H_{012}^2}{2e^2}\,, \tag{2.13}$$

the energy density of the state Ψ_r is

$$E_r = \frac{e^2 r^2}{2}\,. \tag{2.14}$$

Now we add to the Lagrangian an extra term

$$L'_H = -\frac{\theta}{2\pi}\int d^3x H_{012}\,, \tag{2.15}$$

analogous to the theta term in two dimensional QED. This does not affect the quantization, the construction of the physical states Ψ_r, or the Hamiltonian (2.13). However, it adds a constant to Π, which is now $\Pi = H_{012}/e^2 - \theta/2\pi$. Thus, the three-form field strength is now $H_{012} = e^2(r + \frac{\theta}{2\pi})$, and the state Ψ_r now has energy

$$E_r(\theta) = \frac{e^2\left(r + \frac{\theta}{2\pi}\right)^2}{2}\,. \tag{2.16}$$

(Note that, as in [10], under $\theta \to \theta + 2\pi$, there is a monodromy $E_r \to E_{r+1}$. This reflects the fact that $\theta \to \theta + 2\pi$ leaves the theory invariant if accompanied by a gauge transformation that maps $\Psi_r \to \Psi_{r+1}$.)

Now, going back to our ten-dimensional problem, the interaction (2.6) that we used in finite volume reduces on $\mathbf{R}^3$ (after integrating over $D_2 \times \mathbf{S}^5$) to the term (2.15) with

$$\theta = -N\alpha\,. \tag{2.17}$$

The symmetry under $\theta \to \theta + 2\pi$ thus becomes the expected topological symmetry

$$\alpha \to \alpha + \frac{2\pi}{N}\,. \tag{2.18}$$

More explicitly, the energy of the state Ψ_r becomes

$$E_r(\alpha) = \frac{e^2\left(r - \frac{N\alpha}{2\pi}\right)^2}{2}\,. \tag{2.19}$$

This formula is clearly invariant under (2.18) (if one allows for the possibility of changing r) but not under any additional shifts of α. The effective potential for α is

$$V(\alpha) = \min_r E_r(\alpha) = \min_r \frac{e^2\left(r - \frac{N\alpha}{2\pi}\right)^2}{2}. \tag{2.20}$$

This function vanishes precisely if $\alpha = 2\pi r/N$ for some integer r. Thus, the topological symmetry is spontaneously broken, and there are precisely the N vacua required by this symmetry breaking.

Next, let us describe the domain walls between adjacent vacua. Such a domain wall should look macroscopically like a two-dimensional surface $\Sigma \subset \mathbf{R}^3$. Since r jumps by one unit in going between adjacent vacua, the domain wall has the property that as one crosses it, the value of H_{012} jumps by one unit. A D-string world-volume has precisely this property. (Note that in $2+1$ dimensions, a domain wall is in fact a string!) The domain wall of the topological symmetry is thus a D-string of world-volume $\Sigma \times P \times Q \subset \mathbf{R}^3 \times D_2 \times \mathbf{S}^5$, with P and Q being points in D_2 and $\mathbf{S}^5$, respectively. In particular, the energy density of the domain wall is of order N in the 't Hooft large N limit, and not of order N^2 as one would have expected if the domain wall were a soliton in the large N effective field theory. The role of the D-string as a domain wall is somewhat like what was proposed in [11], except that in that discussion there was no analog of the α field.

2.3. Strings ending on domain walls and duality

For a final comment on the $SU(N)$ gauge theory, we note the following. In $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbf{R}^3 \times \mathbf{S}^1$, temporal Wilson lines have a vacuum expectation value associated with spontaneous breaking of the topological symmetry. Meanwhile, spatial Wilson loops

$$\mathrm{Tr} P \exp \oint_C A\,, \tag{2.21}$$

with C a large circle in $\mathbf{R}^3$ (at a specified point in $\mathbf{S}^1$) exhibit an area law, a phenomenon that is equivalent to confinement in the low energy Yang-Mills theory on $\mathbf{R}^3$. As discussed in [4] (using the framework of [8, 9]), the string or *flux tube* associated with this confinement is simply the elementary Type IIB string.

Since elementary strings can end on D-branes, it follows that the flux tubes associated with confinement can terminate on the domain walls that separate different vacua. This is reminiscent of the behavior found in [12] for chiral domain walls in $\mathcal{N} = 1$ super Yang-Mills theory in four dimensions, on which flux tubes associated with confinement can end.

Now consider the $\tau \to -1/\tau$ duality symmetry of the underlying Type IIB theory. In the $\mathcal{N} = 4$ super Yang-Mills theory on $\mathbf{R}^3 \times \mathbf{S}^1$, this symmetry exchanges Wilson loops with 't Hooft loops. Since this symmetry also exchanges fundamental strings with D-strings, it exchanges the flux tubes associated with confinement with the domain walls associated with spontaneous breaking of the topological symmetry.

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In fact, the existence of an $SL(2, \mathbf{Z})$ duality symmetry in this case suggests a more symmetric view of the different strings in the theory. In addition to the angular variable α which we associated with the $\mathbf{Z}_N$ topological symmetry, there is a similar angular variable $\beta = \int_{D_2} B_{RR}$, which may be associated with a magnetic counterpart of the $\mathbf{Z}_N$ symmetry discussed above. Arguments similar to the ones above show that there is a *magnetic* topological symmetry $\beta \to \beta + 2\pi/N$, and that the 't Hooft line may be viewed as an order parameter for the breaking of this $\mathbf{Z}_N$ symmetry. The *magnetic* $\mathbf{Z}_N$ symmetry is spontaneously broken in the finite temperature theory (at infinite volume), and the fundamental string worldsheets serve as domain walls for the magnetic $\mathbf{Z}_N$ topological symmetry.

Altogether, then, there is in fact a $\mathbf{Z}_N \times \mathbf{Z}_N$ topological symmetry. The finite temperature theory has N^2 different vacua, corresponding to the possible values of α and β. The theory includes (p, q) strings (for relatively prime p and q) that serve as domain walls; upon crossing such a domain wall, α changes by $\frac{2\pi q}{N}$ and β changes by $\frac{2\pi p}{N}$. A (p_1, q_1) domain wall and a (p_2, q_2) domain wall can merge into a (p_1+p_2, q_1+q_2) domain wall; in the M theory description of the type IIB string theory (which was described for this type of vacua in [13]) such string intersections become continuous membrane surfaces [14].

3. "Topological symmetry" of the A_{N-1} $(0, 2)$ theory in six dimensions

In six dimensions, there is a family of superconformal theories with $(0, 2)$ supersymmetry and an $A - D - E$ classification. The A_{N-1} theory is believed [1] to be equivalent to M-theory on $AdS_7 \times \mathbf{S}^4$, with N units of four-form flux on $\mathbf{S}^4$. In other words, if C is the three-form potential and $G = dC$, then to get the A_{N-1} theory, one wants

$$\int_{\mathbf{S}^4} \frac{G}{2\pi} = N\,. \tag{3.1}$$

If the A_{N-1} theory is compactified on a circle, we get a five-dimensional theory with $SU(N)$ gauge group at low energies. Upon further compactification on a second circle, a $\mathbf{Z}_N$ topological symmetry arises which is associated with the center of this $SU(N)$ gauge theory. So, the A_{N-1} theory compactified on $\mathbf{S}^1 \times \mathbf{S}^1 = \mathbf{T}^2$ should have a $\mathbf{Z}_N$ global symmetry. Here we will use the AdS-CFT correspondence to demonstrate a stronger result: the $(0, 2)$ theory actually acquires a $\mathbf{Z}_N$ global symmetry if it is compactified on *any* Riemann surface F. This statement will hopefully serve as a clue to the nature of the $(0, 2)$ theory.

We consider the $(0, 2)$ theory on $\mathbf{S}^4 \times F$ (or, taking the volume of $\mathbf{S}^4$ to infinity, on $\mathbf{R}^4 \times F$). According to the general AdS-CFT correspondence, to study the $(0, 2)$ theory on $\mathbf{S}^4 \times F$, we must find a seven-dimensional Einstein manifold X, of negative curvature, with $\mathbf{S}^4 \times F$ at *conformal infinity.* Then we consider M-theory on $X \times Y$,

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where $Y = \mathbf{S}^4$, and sum over the contributions for different choices of X.[4] (The reason for denoting the second factor of $X \times \mathbf{S}^4$ as Y is to avoid confusion with the $\mathbf{S}^4$ factor in the spacetime $\mathbf{S}^4 \times F$ of the $(0,2)$ model.) For large N this sum will presumably be dominated for generic values of the parameters by a single manifold, as in the previous section.

Rather than Wilson lines, the $(0,2)$ theory has two-surface observables. In the free $(0,2)$ theory with a single tensor multiplet including a 2-form field B (with self-dual field strength) these observables are of the form $\exp i \int_\Sigma B$. But the generalization of this expression to the interacting A_{N-1} theory is unknown.[5] In the context of the AdS/CFT correspondence, the observable associated with a closed oriented two-surface $\Sigma \subset \mathbf{S}^4 \times F$ is computed by summing over configurations in which, in $X \times Y$, there is a membrane whose worldvolume W has Σ for boundary. For P a point in $\mathbf{S}^4$, we let $\Sigma(P) = P \times F$, and we denote the observable associated with $\Sigma(P)$ as $W(P)$. We will demonstrate the $\mathbf{Z}_N$ global symmetry by showing that expectation values

$$\langle W(P_1)W(P_2)\ldots W(P_k)\rangle \tag{3.2}$$

vanish unless k is a multiple of N. According to Poincaré-Lefschetz duality, the subspace of $H_*(\mathbf{S}^4 \times F)$ consisting of homology classes that are boundaries in X is a *maximal isotropic subspace.* It follows that the possible X's are of two kinds:

(1) $\mathbf{S}^4$ is a boundary in X, but F is not. An example is $X = D_5 \times F$, where the boundary of D_5 is $\mathbf{S}^4$.

(2) F is a boundary in X, but $\mathbf{S}^4$ is not. An example is $X = \mathbf{S}^4 \times D_3$, where the boundary of D_3 is F.

We want to show that every X has the property that M-theory on $X \times Y$ contributes to (3.2) only if k is a multiple of N.

For X of type (1), the surfaces $\Sigma(P_i)$ are not boundaries in $X \times Y$, so to get a nonzero contribution to (3.2), we need a *baryon vertex* on which the membranes that begin on $\Sigma(P_i)$ can end.

This can be found as follows. For X of type (1), a fivebrane wrapped on $F \times Y$ is stable. It serves as a baryon vertex for the following reason. Let T be the self-dual threeform field on the fivebrane. Let W be the union of all membrane worldvolumes, and – bearing in mind that membranes can end on fivebranes – let ∂W be the Riemann surface in the fivebrane worldvolume that consists of boundaries of membranes. Since the boundaries of membranes in fivebranes are charged under T, one has

$$dT = G - 2\pi\delta(\partial W)\,. \tag{3.3}$$

[4] Several possible generalizations may be necessary. For example, X might contain branes or other singularities, or it may be incorrect in general to assume that the M-theory spacetime is a product $X \times \mathbf{S}^4$. Nevertheless, the case stated in the text seems illustrative.

[5] The generalization of loop equations to this type of surface observables was discussed in [15].

Because of (3.1), this equation can be solved for T precisely if ∂W is homologous to N copies of F. Thus, the fivebrane on $F \times Y$ contributes to (3.2) precisely if $k = N$. By taking several such fivebranes, one gets contributions for k any multiple of N.

For an X of type (2), we have instead that the individual $\Sigma(P_i)$ are boundaries in X, so at first sight it appears that $\langle W(P_i)\rangle$ is nonzero for each i. However, for X of type (2), F is the boundary of a three-cycle B in X. At no cost in action, one can turn on a C field on $X \times Y$ with nonzero

$$\alpha = \int_B C \tag{3.4}$$

but $G = 0$. The matrix element (3.2) receives its contribution from membranes wrapped on a cycle such as B, so the coupling of C to membranes gives an α-dependent factor $e^{ik\alpha}$ to this matrix element. If this were the only α-dependent factor, then integration over α would cause the matrix element to vanish for all k.

To get a nonzero result for suitable k, we proceed just as we did in section 2 for the case of the $\mathcal{N} = 4$ theory in four dimensions. On $\mathbf{S}^4 \times F$, we pick a four-form G with

$$\int_{\mathbf{S}^4} \frac{G}{2\pi} = s\,, \tag{3.5}$$

for some $s \in \mathbf{Z}$. Because X is of type (2), $G/2\pi$ extends over X as a closed fourform with integral periods. By minimizing its action (an important detail is that the action of G is in fact finite!) subject to the condition (3.5), G can be turned into a harmonic four-form that represents an instanton in M-theory on $X \times Y$. Because of the existence of an interaction in M-theory of the form

$$-\frac{1}{6(2\pi)^2}\int C \wedge G \wedge G\,, \tag{3.6}$$

and using (3.1), the amplitude for this instanton has a phase factor $e^{-isN\alpha}$. By choosing $s = k/N$, one cancels the factor $e^{ik\alpha}$ and gets a nonzero contribution to (3.2) whenever k is a multiple of N. Thus, again we find that the A_{N-1} $(0,2)$ theory compactified on a Riemann surface has a $\mathbf{Z}_N$ symmetry.

3.1. Behavior for infinite volume

As in section 2, we can similarly study the $(0,2)$ theory on $\mathbf{R}^4 \times F$. For this, one must study M-theory on spacetimes $\mathbf{R}^4 \times D_3 \times Y$ for some D_3. The key point is to study the dynamics of the field G_{0123} on $\mathbf{R}^4$. An analysis as in section 2, using the interaction term (3.6), gives an effective potential $V(\alpha)$ of the same form as in (2.20). In particular, the $\mathbf{Z}_N$ symmetry is spontaneously broken; there are N vacua, at $\alpha = 2\pi r/N$, $r = 0, 1, \ldots, N-1$. The domain walls between successive vacua are simply the membranes of M-theory.

One can also consider the observable $W(\Sigma)$ for Σ a surface in $\mathbf{R}^4$ (times a point in F). This is analogous to the *spatial Wilson loop* of the $\mathcal{N} = 4$ theory on $\mathbf{R}^3 \times \mathbf{S}^1$.

The expectation value $\langle W(\Sigma)\rangle$ shows a *volume law* – it vanishes exponentially with the volume of a minimal three-surface of boundary Σ, because of the appearance of a *flux surface* (analogous to flux tubes in confining gauge theories) spanning Σ. This can be argued just as in [4] for the spatial Wilson loops of the $\mathcal{N}=4$ theory. The *flux surfaces* are M-theory membranes, so in this case the flux surfaces associated with *confinement* also double as *domain walls* for spontaneous breaking of the topological symmetry. These flux surfaces can end on the domain walls, since there is no trouble in joining two membranes smoothly. The fact that, unlike in the previous section, we find only one $\mathbf{Z}_N$ factor may be viewed as the generalization to the interacting theories of the self-duality of the field strength of the 2-form B of the free $(0,2)$ theory.

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